Rational homotopy of symmetric products and Spaces of finite subsets

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Abstract. We describe Sullivan models of the symmetric products of spaces and of some symmetric fat diagonals. From the determination of models of some spaces of finite subsets, we verify a Tuffley's conjecture in low ranges for rational spaces. Extending a theorem of Handel, we prove also the triviality of the inclusion of a space X in the space of $(n+2)$-th finite subsets of X, when X is of Lusternik-Schnirelmann category less than or equal to n.

The properties of configuration spaces of a space X have deserved many studies where the use of algebraic models was limited by the fact that the homotopy type of configuration spaces of points in X is not a homotopy invariant of X. The spaces of finite subsets, $\exp^n X$, of a space X are a substitute of configuration spaces where this anomaly does not exist any more, see [**14**]. They are related to configuration spaces by the fact that the cofiber of the canonical inclusion $\exp^{n-1} X \hookrightarrow \exp^n X$ is the one point compactification of the configuration space of unordered n-uples of distinct points of a compact space X, see [**14**, Proposition 2.3].

Recall that the n-th finite subsets space, $\exp^n X$, of a topological (non-empty) space X is the space of non-empty subsets of size at most n, topologised as the quotient of X^n by the surjective map $X^n \to \exp^n X$, $(x_1, \ldots, x_n) \mapsto \{x_1\} \cup \cdots \cup \{x_n\}$. The introduction of the spaces of finite subsets of a space is due to K. Borsuk and S. Ulam [**3**], [**2**]. Their study was continued by R. Bott [**4**] and more recently by D. Handel [**14**], R. Biro [**1**], C. Tuffley [**26**], [**27**], [**28**], J. Mostovoy [**19**], [**20**], S. Rose [**22**], S. Kallel and D. Sjerve [**16**], [**17**]. As $X \mapsto \exp^n X$ are homotopy functors, the elaboration of rational models for them is an open challenge and we develop here the first steps in this direction.

First, we quote a non rational result inspired by a theorem of Handel. If X is pointed, we denote by $\exp^n_* X$ the subspace of $\exp^n X$ formed from the subsets that contain the basepoint and by $\iota^n_X \colon X \to \exp^n_* X$ the map that adjoins the basepoint to each subset. This space $\exp^n_* X$ is often used as a first step in the study of $\exp^n X$. For any arc-connected pointed space $(X, *)$, Handel shows in ([**14**, Theorem 4.1 et Theorem 4.2]) that the two maps $\pi_j(\exp^n_* X) \to \pi_j(\exp^{2n-1}_* X)$ and $\pi_j(\exp^n X) \to \pi_j(\exp^{2n+1} X)$, induced by the inclusion, are the zero map. From these results, Handel deduces that the space $\exp^\infty X = \cup_{n\geq 1} \exp^n X$ is weakly

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contractible. In this paper, we show that the triviality is also at the level of some inclusions between finite subset spaces.

Theorem 1. *If the Lusternik-Schnirelmann category of a pointed CW-complex* $(X,*)$ *is less than or equal to* n*, the following inclusions are homotopically trivial.*

(1) $\iota_X^{n+2}\colon X \to \exp_*^{n+2} X$,
(2) $X \to \exp^{n+2} X$,
(3) $\exp_*^{k+1} X \to \exp_*^{(n+1)k+1} X$.

The rest of the paper is concerned with Sullivan models for which we refer to [**9**], [**10**] or [**25**]. As $\exp^n X$ can be inductively built from the symmetric product $\mathrm{Sp}^n X$ and its fat diagonal $\mathbb{\Delta}_{\mathcal{S}}^n X$ (see Definition 3.1 and the pushout 4.2), we begin by a study of models of those spaces. First, we recall some known basic facts for the models of spaces with an action of a finite group. We apply it to symmetric products and prove the next result.

Theorem 2. *Let* (A,d) *be a connected model of a connected space* X*. Then the cdga* $\wedge^n(A,d)$ *is a model of* $\mathrm{Sp}^n X$*, with a multiplicative law on* $\wedge^n A$ *given by*

$$(a_1 \wedge \cdots \wedge a_n) * (b_1 \wedge \cdots \wedge b_n) = \sum_{\sigma\in\mathcal{S}_n} \pm (a_1 \bullet b_{\sigma(1)}) \wedge \cdots \wedge (a_n \bullet b_{\sigma(n)}),$$

where $\bullet$ *is the multiplicative law of* A*,* $\mathcal{S}_n$ *is the symmetric group and* $\pm$ *means the Koszul sign.*

From this model, we deduce the (additive) rational cohomology of the symmetric product described by D. Zagier in [**30**]. For instance, we get a short proof for the determination of the Poincaré polynomial of $\mathrm{Sp}^n X$ in terms of the Betti numbers of X, previously obtained by I.G. MacDonald in [**18**]. We give a description of the rational homotopy type of $\mathrm{Sp}^n \Sigma X$ and compute a model of $\mathrm{Sp}^2 \mathbb{C}P^2$, see Proposition 2.5 and Example 2.6. We get also an inductive construction of the fat diagonals $\mathbb{\Delta}_{\mathcal{S}}^n X$ (see Proposition 3.2) which allows the determination of $\mathbb{\Delta}_{\mathcal{S}}^n S^{2k+1}$ for any n. This works ends with the determination of the rational homotopy type of $\exp^3 \Sigma X$ and $\exp^4 \Sigma X$. An important conjecture of the theory, due to Tuffley ([**28**]), states as follows.

Tuffley's Conjecture. *If* X *is an* r*-connected CW-complex, the space* $\exp^n X$ *is* $(n+r-2)$*-connected.*

In fact, Tuffley proves that $\exp^n X$ is $(n-2)$-connected if X is connected and $(n-1)$-connected if X is simply connected. He shows also that it is sufficient to prove the conjecture for a wedge of spheres ([**29**, Theorem 2]). From that observation and from our results, we deduce that the Tuffley's conjecture is true for the rationalisation of $\exp^3 X$ and $\exp^4 X$. Tuffley's conjecture has also been verified for $n=3$ in recent work of S. Kallel and D. Sjerve, [**17**].

Our program is carried out in Sections 1-6 below, whose headings are self-explanatory.

Contents

1. Algebraic models of G-spaces with G a finite group

In this section, we recall some basic facts on Sullivan models of spaces on which acts a finite group G.

By definition, a G-dga is a differential graded algebra (A, d_A), with $H^0(A, d_A) = \mathbb{Q}$, on which G acts by dga maps. If the algebra A is commutative graded, we use the expression G-cdga. The invariant subspace $(A, d_A)^G$ of a G-dga defines a sub-dga of (A, d_A) and if $f\colon (A, d_A) \to (B, d_B)$ is a G-equivariant quasi-isomorphism, we have the following well-known properties (see [**11**, Section 1] for instance):

- $f^G\colon (A, d_A)^G \to (B, d_B)^G$ is also a quasi-isomorphism,
- $H((A, d_A)^G) = (H(A, d_A))^G$.

If V is a graded $\mathbb{Q}$-vector space, we denote by $T(V)$ the free associative graded algebra on V and by $\wedge V$ the free commutative graded algebra on V. Any G-cdga, (A, d_A), admits a minimal model

$$\varphi\colon (\wedge V, d) \to (A, d_A)$$

with an action of G on $(\wedge V, d)$ making the map φ G-equivariant, see [**13**]. This model is unique up to G-isomorphisms and we call it the minimal G-model of (A, d_A). More generally, a *G-model* of (A, d_A) is any G-cdga having the same minimal G-model as (A, d_A).

We apply these algebraic data to spaces with a G-action. Let X be a simplicial complex with a (simplicial) action of G. Recall from G. Bredon ([**5**, Page 115]) that the action is *regular* if, for any $g_0, \dots, g_n$ in G and simplices $(v_0, \dots, v_n)$, $(g_0v_0, \dots, g_nv_n)$ of X, there exists an element $g \in G$ such that $gv_i = g_iv_i$ for all i. By [**5**, Proposition 1.1, Page 116], the induced action on the second barycentric subdivision is always regular. Here, *by a G-space, we mean a connected simplicial complex on which G acts regularly.*

Denote by $C(X)$ the oriented chain complex of a G-space X and observe that $C(X)$ is a module over the group ring $\mathbb{Z}[G]$ of G. The canonical simplicial map $\rho\colon X \to X/G$ induces $\rho_*\colon C(X) \to C(X/G)$. Define now $\sigma\colon C(X) \to C(X)$, $c \mapsto \sum_{g\in G} gc$. One has $\mathrm{K}er\,\sigma = \mathrm{K}er\,\rho_*$. Therefore σ induces $\overline{\sigma}$

$$\begin{tikzcd} C(X) \arrow[r, "\sigma"] \arrow[d, "\rho_*"'] & C(X) \\ C(X/G) \arrow[ur, "\overline{\sigma}"'] & \end{tikzcd}$$

such that $\overline{\sigma} \circ \rho_* = \sigma$. Bredon proves the next result.

Proposition 1.1 ([**5**, Page 120]). *Let $\Bbbk$ be a field of characteristic that does not divide the order of G and let X be a G-space. Then there are isomorphisms*

$$C_*(X/G; \Bbbk) \cong C_*(X; \Bbbk)/G \text{ and } C^*(X/G; \Bbbk) \cong C^*(X; \Bbbk)^G\,.$$

When X is a G-space, the (finite) group G acts on the Sullivan algebra of PL-forms on X, $A_{PL}(X)$, [**24**]. As in Proposition 1.1, the cdga's $A_{PL}(X/G)$ and $A_{PL}(X)^G$ are isomorphic and this isomorphism gives models of the quotient X/G from G-models of X, see [**11**, Proposition 7].

Proposition 1.2. *Let (A, d_A) be a G-model of the G-space X, then the cdga $(A, d_A)^G$ is a model of X/G.*

2. Symmetric products and Proof of Theorem 2

In this section, we describe Sullivan models of symmetric products of a space.

Definition 2.1. Let X be a space. The *symmetric product,* $\mathrm{Sp}^n X$, is the quotient of the product X^n by the action of the symmetric group $\mathcal{S}_n$,

$$\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

with $\sigma \in \mathcal{S}_n$ and $(x_1, \dots, x_n) \in X^n$. We denote by $\langle x_1, \dots, x_n\rangle \in \mathrm{Sp}^n X$ the class associated to the element $(x_1, \dots, x_n) \in X^n$ and by $\rho_n \colon X^n \to \mathrm{Sp}^n X$ the canonical projection.

As first examples of symmetric products, recall that $\mathrm{Sp}^n S^1$ is homotopy equivalent to S^1, $\mathrm{Sp}^n S^2$ diffeomorphic to $\mathbb{C}P^n$ and $\mathrm{Sp}^n \mathbb{R}P^2$ diffeomorphic to $\mathbb{R}P^{2n}$, see [**21**], [**19**] and [**8**] for more details and historical comments.

If the space X is pointed by $*$, the adding of $*$ gives an inclusion of $\mathrm{Sp}^{n-1} X$ in $\mathrm{Sp}^n X$. By definition, the infinite symmetric product $\mathrm{Sp}^\infty X$ is the direct limit of the spaces $\mathrm{Sp}^n X$. If X is connected, this infinite product is a product of Eilenberg-McLane spaces, $\mathrm{Sp}^\infty(X, *) \simeq \prod_i K(\tilde{H}_i(X;\mathbb{Z}), i)$, see [**7**]. By convention, we set $\mathrm{Sp}^0 X = *$.

Proof of Theorem 2. For any cdga (A, d), and any integer $n \geq 1$, the symmetric group $\mathcal{S}_n$ acts on the tensor product $\otimes^n(A, d)$ by permuting the factors,

$$\sigma(a_1 \otimes \dots \otimes a_n) = \pm(a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n)}),$$

where the sign comes from the ordinary rule of permutation of graded objects. If (A, d) is a model of X, as a direct consequence of Proposition 1.2, the cdga $(\otimes^n(A, d))^{\mathcal{S}_n}$ is a model of $\mathrm{Sp}^n X$.

To make the structure of the cdga $(\otimes^n(A, d))^{\mathcal{S}_n}$ more precise, recall the existence of an isomorphism $\wedge^n A \cong (\otimes^n A)^{\mathcal{S}_n}$ defined by

$$a_1 \wedge \dots \wedge a_n \mapsto \sum_{\sigma \in \mathcal{S}_n} \pm a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n)}.$$

Denote by $\bullet$ the multiplicative law of A. The isomorphism above transforms the product on $(\otimes^n(A, d))^{\mathcal{S}_n}$ to the following law of algebra on $\wedge^n A$,

$$(a_1 \wedge \dots \wedge a_n) * (b_1 \wedge \dots \wedge b_n) = \sum_{\sigma \in \mathcal{S}_n} \pm(a_1 \bullet b_{\sigma(1)}) \wedge \dots \wedge (a_n \bullet b_{\sigma(n)}),$$

with $\frac{1}{n!}(1 \wedge \dots \wedge 1)$ as identity. $\square$

Theorem 2 implies that the cohomology vector space of $\mathrm{Sp}^n X$ depends only on the cohomology vector space of X and gives the Poincaré polynomial of a symmetric product, a formula due to I.G. Macdonald.

COROLLARY 2.2. [**18**] *Denote by* $(b_i)_{i\geq 0}$ *the Betti numbers of a space* X. *Then the Poincaré polynomial of the space* $\mathrm{Sp}^n X$ *is the coefficient of* t^n *in the power series associated to:*

$$\frac{(1+xt)^{b_1}(1+x^3t)^{b_3}\dots}{(1-t)^{b_0}(1-x^2t)^{b_2}(1-x^4t)^{b_4}\dots} = \frac{\prod_{i=0}(1+x^{2i+1}t)^{b_{2i+1}}}{\prod_{i=0}(1-x^{2i}t)^{b_{2i}}}.$$

PROOF. This is a direct consequence of the well-known Poincaré polynomial of the algebra $\wedge a$ where a is a generator of odd or even degree. □

Theorem 2 implies also immediately the following property.

COROLLARY 2.3. *If* X *is formal, the symmetric product* $\mathrm{Sp}^n X$ *is formal, for any* $n \geq 1$.

In the particular case of a suspension, the model of Theorem 2 can be expressed in a simple manner.

COROLLARY 2.4. *If* X *is a suspension having rational cohomology* H, *we have an isomorphism of algebras* $\psi\colon (\wedge H^+/\wedge^{>n} H^+, 0) \to (\wedge^n H, 0)$, *defined by* $\psi(1) = \dfrac{1}{n!}(1\wedge\dots\wedge 1)$ *and* $\psi(x) = \dfrac{x\wedge 1\wedge\dots\wedge 1}{(n-1)!}$ *if* $x \in H^+$. *In particular, the cdga* $(\wedge H^+/\wedge^{>n} H^+, 0)$ *is a model of* $\mathrm{Sp}^n X$.

PROOF. As $H = \mathbb{Q}\oplus H^+$ with $H^+ = \oplus_{i\geq 1} H^i$, the algebra $\wedge^n H$ is isomorphic, as vector space, to $\wedge^{\leq n} H^+ \cong \wedge H^+/\wedge^{>n} H^+$. The product being null on H^+, the law of algebra of the quotient $\wedge H^+/\wedge^{>n} H^+$ corresponds, by ψ, to the product $*$ defined on $\wedge^n H$ in Theorem 2 and the result follows. □

In the case of a sphere S^p, Corollary 2.4 gives $(\wedge a/\wedge^{>n} a, 0)$, with a of degree p, as model of $\mathrm{Sp}^n S^p$. More precisely,

- if $p = 2k+1$, $\mathbb{Q}\oplus a\mathbb{Q}$ is a model of S^{2k+1} and we have $\mathrm{Sp}^n S^{2k+1} \simeq_{\mathbb{Q}} S^{2k+1}$, for any $n \geq 1$;
- if $p = 2k$, we denote by $P^n S^{2k}$ the rational space having $(\wedge a/(a^{n+1}), 0)$ as model, with a of degree $2k$. (Observe that $P^n S^2 = \mathbb{C}P^n$.) For any $n \geq 1$, we have $\mathrm{Sp}^n S^{2k} \simeq_{\mathbb{Q}} P^n S^{2k}$ and $P^\infty S^{2k} = K(\mathbb{Q}, 2k)$.

The next result gives the precise rational homotopy type of the symmetric product of a general suspension. To state it, we first define a filtration on a product of filtered spaces, as follows. If the spaces X and Y are given with an increasing filtration $X^{(0)} = * \subset X^{(1)} \subset \dots \subset X^{(l-1)} \subset X = X^{(l)}$, $Y^{(0)} = * \subset Y^{(1)} \subset \dots \subset Y^{(k-1)} \subset Y = Y^{(k)}$, the product space is given with the filtration

$$(X\times Y)^{(i)} = \bigcup_{i_1+i_2=i} X^{(i_1)} \times Y^{(i_2)}.$$

We endow the odd sphere with the trivial filtration, $(S^{2i+1})^{(1)} = S^{2i+1}$, and the space $P^n S^{2k}$ with the filtration defined by: $(P^n S^{2k})^{(l)}$ is the $2kl$-skeleton of $P^n S^{2k}$.

PROPOSITION 2.5. *Let* X *be a suspension of cohomology* $H = H^*(X;\mathbb{Q})$. *Let* $(\alpha_1,\dots,\alpha_l)$ *be a basis of* H^{odd} *and* $(\beta_1,\dots,\beta_k)$ *be a basis of* H^{even}. *Then, the commutative graded algebra* $(\wedge H^+/\wedge^{>n} H^+, 0)$ *is a model of* $\mathrm{Sp}^n X$ *and we have*

$$\mathrm{Sp}^n(X) \simeq_{\mathbb{Q}} \left(\prod_{i=1}^{l} S^{|\alpha_i|} \times \prod_{j=1}^{k} P^n S^{|\beta_j|}\right)^{(n)}.$$

Moreover a model of the projection $\rho_{n,q}\colon X^n\times \mathrm{Sp}^q X\to \mathrm{Sp}^{n+q} X$ *is the cdga's map* $\varphi_{n,q}\colon (\wedge H^+/\wedge^{>n+q} H^+,0)\to (H^{\otimes n}\otimes (\wedge H^+/\wedge^{>n} H^+),0)$, *which sends* $x\in H^+$ *on* $(x\otimes 1\otimes\ldots\otimes 1)\otimes 1+\cdots+(1\otimes\ldots\otimes 1\otimes x)\otimes 1+(1\otimes\ldots\otimes 1)\otimes x$.

For instance, Proposition 2.5 implies $\mathrm{Sp}^n(S^3\vee S^5)\simeq_{\mathbb{Q}} S^3\times S^5$ for any $n\geq 2$, $\mathrm{Sp}^n(S^3\vee S^5\vee S^7)\simeq_{\mathbb{Q}} (S^3\times S^5\times S^7)$ for any $n\geq 3$ and $\mathrm{Sp}^2(S^3\vee S^5\vee S^7)$ has the rational homotopy type of the fat wedge of (S^3,S^5,S^7). Observe also that Proposition 2.5 is coherent with results of Snaith and Ucci, see [**23**, Proposition 2.2] and of B. W. Ong on $\mathrm{Sp}^n(\vee S^1)$, [**21**].

Proof. The first part of the statement comes directly from Corollary 2.4. For the second part, we start with the canonical surjection $\rho_n=\rho_{n,0}\colon X^n\to \mathrm{Sp}^n X$, of model the canonical inclusion $(H^{\otimes n})^{\mathcal{S}_n}\hookrightarrow H^{\otimes n}$. Now we replace $(H^{\otimes n})^{\mathcal{S}_n}$ by the cga $\wedge H^+/\wedge^{>n} H^+$ as in Corollary 2.4. The sequence of isomorphisms

$$\wedge H^+/\wedge^{>n} H^+ \xrightarrow{\cong} \wedge^n H \xrightarrow{\cong} (H^{\otimes n})^{\mathcal{S}_n}$$

is defined by $x \rightarrowtail \dfrac{x\wedge 1\wedge\ldots\wedge 1}{(n-1)!} \rightarrowtail x\otimes 1\otimes\ldots\otimes 1+\cdots+1\otimes\ldots\otimes 1\otimes x.$

Composed with these isomorphisms, the inclusion $(H^{\otimes n})^{\mathcal{S}_n}\hookrightarrow H^{\otimes n}$ becomes $\varphi_{n,0}\colon \wedge H^+/\wedge^{>n} H^+\to H^{\otimes n}$, $\varphi_{n,0}(x)=x\otimes 1\otimes\ldots\otimes 1+\cdots+1\otimes\ldots\otimes x$.

In the general case of the projection $\rho_{n,q}\colon X^n\times \mathrm{Sp}^q X\to \mathrm{Sp}^{n+q} X$, we have a commutative diagram

$$\begin{tikzcd}
X^{n+q} \arrow[r, equal] \arrow[d, "\rho_{n+q}"'] & X^n\times X^q \arrow[d, "1\times\rho_q"] \\
\mathrm{Sp}^{n+q} X & X^n\times \mathrm{Sp}^q X \arrow[l, "\rho_{n,q}"']
\end{tikzcd}$$

which induces in cohomology the diagram

$$\begin{tikzcd}
H^{\otimes(n+q)} \arrow[r, equal] & H^{\otimes n}\otimes H^{\otimes q} \\
\wedge H^+/\wedge^{>n+q} H^+ \arrow[u, "\varphi_{n+q,0}"] \arrow[r, "\varphi_{n,q}"] & H^{\otimes n}\otimes(\wedge H^+/\wedge^{>q} H^+). \arrow[u, "1\otimes\varphi_{q,0}"']
\end{tikzcd}$$

Since $\varphi_{n+q,0}$ and $\varphi_{q,0}$ are injective, the morphism $\varphi_{n,q}$ is given on $x\in H^+$ by $\varphi_{n,q}(x)=(x\otimes 1\otimes\ldots\otimes 1)\otimes 1+\cdots+(1\otimes\ldots\otimes 1\otimes x)\otimes 1+(1\otimes\ldots\otimes 1)\otimes x$. $\square$

Example 2.6. We study now $\mathrm{Sp}^2(\mathbb{C}P^2)$. Its vector space of cohomology is $\wedge^2(1,\beta_1,\beta_2)$ with $|\beta_1|=2$ and $|\beta_2|=4$. We denote $a=\beta_1\wedge 1$, $b=\beta_2\wedge 1$, $c=\beta_1\wedge\beta_1$, $e=\beta_1\wedge\beta_2$, $f=\beta_2\wedge\beta_2$. The law of algebra is given by Theorem 2 as follows, $a^2=(\beta_1\wedge 1)*(\beta_1\wedge 1)=(\beta_1\bullet\beta_1)\wedge 1+\beta_1\wedge\beta_1=\beta_2\wedge 1+\beta_1\wedge\beta_1=b+c$.

We compute similarly the other products and get $e=ab$, $ac=2e$, $ae=b^2=f$, $c^2=2f$ and zero for the other ones. The algebra structure can be described by $H^*(\mathrm{Sp}^2(\mathbb{C}P^2);\mathbb{Q})=\{a,b,a^2,a^3,a^4\}\mathbb{Q}$, with $a^3=3ab$ and $b^2=a^2b=a^4/3$.

The projection $\{a,b,a^2,a^3,a^4\}\mathbb{Q}\to b\mathbb{Q}$ gives a map $S^4\to \mathrm{Sp}^2\,\mathbb{C}P^2$ whose rational homotopy cofiber is $\mathbb{C}P^4$; the associated long exact sequence of this cofibration splits in short ones.

Remark 2.7. If (A,d_A) is a model of X, the canonical inclusion $\mathrm{Sp}^{n-1} X\to \mathrm{Sp}^n X$, obtained by adding a fixed point x_0, has for model the cdga's map, $\wedge^n H\to$

$\wedge^{n-1}H$, defined by $a_1 \wedge \cdots \wedge a_n \mapsto \sum_{i=1}^n \varepsilon(a_i)\ a_1 \wedge \ldots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_n$, where $\varepsilon\colon A \to \mathbb{Q}$ is the augmentation corresponding to $\{x_0\} \hookrightarrow X$. In the case of an odd sphere, the adding of x_0 is clearly a homotopy equivalence, $\mathrm{Sp}^m S^{2k+1} \cong \mathrm{Sp}^{m+1} S^{2k+1}$.

3. Fat diagonal of the symmetric product

This section contains an inductive description of the fat diagonal of a symmetric product and the determination of its rational homotopy type in the case of an odd sphere.

DEFINITION 3.1. The *fat diagonal*, $\mathbb{\Delta}^n_{\mathcal{S}} X$, of the symmetric space $\mathrm{Sp}^n X$ is defined by

$$\mathbb{\Delta}^n_{\mathcal{S}} X = \{\langle x_1, \ldots, x_n\rangle \in \mathrm{Sp}^n(X) \mid \text{there exist } i \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, n\} \text{ such that } i \neq j \text{ and } x_i = x_j\}.$$

The fat diagonal can be easily determined in low ranges, the maps $X \to \mathbb{\Delta}^2_{\mathcal{S}} X$, $x \mapsto \langle x, x\rangle$, and $X \times X \to \mathbb{\Delta}^3_{\mathcal{S}} X$, $(x, y) \mapsto \langle x, x, y\rangle$, being homeomorphisms if X is a finite CW-complex. For the study of the general case, we introduce the subspace $\mathbb{\Delta}^{n,k}_{\mathcal{S}} X$ of $\mathrm{Sp}^n X$, defined as the image of the product $X^k \times X^{n-2k}$ by the map

$$(x_1, \ldots, x_k, y_1, y_2, \ldots, y_{n-2k}) \mapsto (x_1, x_1, x_2, x_2, \ldots, x_k, x_k, y_1, y_2, \ldots, y_{n-2k}).$$

We clearly have $\mathbb{\Delta}^n_{\mathcal{S}} X = \mathbb{\Delta}^{n,1}_{\mathcal{S}} X$, $\mathbb{\Delta}^{2k,k}_{\mathcal{S}} X \simeq \mathrm{Sp}^k X$ and $\mathbb{\Delta}^{2k+1,k}_{\mathcal{S}} X \simeq X \times \mathrm{Sp}^k X$. These diagonal spaces can be constructed inductively as follows.

PROPOSITION 3.2. *We have a homotopy pushout*

$$\begin{tikzcd}
\mathrm{Sp}^k X \times \mathbb{\Delta}^{n-2k}_{\mathcal{S}} X \arrow[r, hook, "g"] \arrow[d, "\varphi"'] & \mathrm{Sp}^k X \times \mathrm{Sp}^{n-2k} X \arrow[d, "\psi"] \\
\mathbb{\Delta}^{n,k+1}_{\mathcal{S}} X \arrow[r, hook, "f"] & \mathbb{\Delta}^{n,k}_{\mathcal{S}} X
\end{tikzcd}$$

where g and f are canonical inclusions, φ and ψ consist to double the first k coordinates.

PROOF. This square is clearly a pushout. As $\mathbb{\Delta}^{n-2k}_{\mathcal{S}} X \hookrightarrow \mathrm{Sp}^{n-2k} X$ is a cofibration, it is also a homotopy pushout. □

This inductive construction allows the determination of the fat diagonal in the case of an odd sphere.

PROPOSITION 3.3. *The symmetric fat diagonals of an* odd-dimensional *sphere, S^n, satisfy the next properties.*

(1) *The canonical inclusions*

$$\mathbb{\Delta}^{2m,m}_{\mathcal{S}} S^n \subset \mathbb{\Delta}^{2m,m-1}_{\mathcal{S}} S^n \subset \cdots \subset \mathbb{\Delta}^{2m,1}_{\mathcal{S}} S^n \subset \mathrm{Sp}^{2m} S^n$$

are rational homotopy equivalences. This implies $\mathbb{\Delta}^{2m,k}_{\mathcal{S}} S^n \simeq_{\mathbb{Q}} S^n$ for all (m, k) with $m \geq k \geq 1$.

(2) *For any $m \geq 1$ and $1 \leq k \leq m$, one has*

$$\mathbb{\Delta}^{2m+1,k}_{\mathcal{S}} S^n \simeq_{\mathbb{Q}} S^n \times (*^{m-k+1} S^n) \simeq S^n \times S^{(n+1)(m+1-k)-1}.$$

Moreover, the inclusion of $\mathbb{\Delta}_{\mathcal{S}}^{2m+1,k+1}\,S^n$ *in* $\mathbb{\Delta}_{\mathcal{S}}^{2m+1,k}\,S^n$ *restricts to the identity on* S^n*.*

(3) *Let* x_0 *be a point in* S^n*. The map* $\delta_{x_0}\colon \mathbb{\Delta}_{\mathcal{S}}^{2m-1,k}\,S^n \to \mathbb{\Delta}_{\mathcal{S}}^{2m+1,k+1}\,S^n$*, obtained by adding* (x_0,x_0)*, is a rational homotopy equivalence, for any* k*,* $1\leq k\leq m-1$*.*

Proof. (1) We work by induction on m. For $m=1$, the inclusion $\mathbb{\Delta}_{\mathcal{S}}^{2}S^n\simeq S^n\to \mathrm{Sp}^2\,S^n$ is a rational homotopy equivalence. Suppose that the result is true for q, $q<m$, i.e. $\mathbb{\Delta}_{\mathcal{S}}^{2m-2i}\,S^n\to \mathrm{Sp}^{2m-2i}\,S^n$ is a rational homotopy equivalence for any i, $1\leq i\leq m$. In the pushout,

$$\begin{tikzcd}
\mathrm{Sp}^i\,S^n\times \mathbb{\Delta}_{\mathcal{S}}^{2m-2i}\,S^n \arrow[r, hook] \arrow[d] & \mathrm{Sp}^i\,S^n\times \mathrm{Sp}^{2m-2i}\,S^n \arrow[d] \\
\mathbb{\Delta}_{\mathcal{S}}^{2m,i+1}\,S^n \arrow[r] & \mathbb{\Delta}_{\mathcal{S}}^{2m,i}\,S^n
\end{tikzcd}$$

the top map is a rational homotopy equivalence. Thus, the bottom map is a rational homotopy equivalence also. The composite of the canonical maps

$$\mathrm{Sp}^m\,S^n\to \mathbb{\Delta}_{\mathcal{S}}^{2m,m}\,S^n\to \mathbb{\Delta}_{\mathcal{S}}^{2m,m-1}\,S^n\to\cdots\to \mathbb{\Delta}_{\mathcal{S}}^{2m}\,S^n\to \mathrm{Sp}^{2m}\,S^n$$

is the square map $\mathrm{Sp}^m\,S^n\to \mathrm{Sp}^{2m}\,S^n$. The commutativity of the diagram

$$\begin{tikzcd}
(S^n)^m \arrow[r, "\text{square}"] \arrow[d, "\rho_m"'] & (S^n)^{2m} \arrow[d, "\rho_{2m}"] \\
\mathrm{Sp}^m\,S^n \arrow[r, "\text{square}"] & \mathrm{Sp}^{2m}\,S^n
\end{tikzcd}$$

shows that the map induced in cohomology by the square map,
$$H^*(\text{square})\colon H^*(\mathrm{Sp}^{2m}\,S^n)=\wedge x\to H^*(\mathrm{Sp}^mS^n)=\wedge x,$$
is the multiplication by 2 and the square map is a rational homotopy equivalence. Therefore, the inclusion $\mathbb{\Delta}_{\mathcal{S}}^{2m}\,S^n\to \mathrm{Sp}^m\,S^n$ is also a rational homotopy equivalence.

(2) As we observed before, the result is true for $k=m$ and we can begin with $k=m-1$. We consider the homotopy pushout

$$\begin{tikzcd}
\mathrm{Sp}^{m-1}\,S^n\times \mathbb{\Delta}_{\mathcal{S}}^{3}\,S^n \arrow[r, "g_1"] \arrow[d, "g_2"'] & \mathrm{Sp}^{m-1}\,S^n\times \mathrm{Sp}^3\,S^n \arrow[d, "g_4"] \\
\mathbb{\Delta}_{\mathcal{S}}^{2m+1,m}\,S^n \arrow[r, hook] & \mathbb{\Delta}_{\mathcal{S}}^{2m+1,m-1}\,S^n,
\end{tikzcd}$$

where g_2 is the square map on the factor $\mathrm{Sp}^{m-1}\,S^n$. With the notation of Proposition 2.5, the map g_2 can be expressed as a product of $\rho_{m-1,1}$ with the identity map:

$$\mathrm{Sp}^{m-1}\,S^n\times \mathbb{\Delta}_{\mathcal{S}}^{3}\,S^n\simeq (\mathrm{Sp}^{m-1}\,S^n)\times S^n\times S^n\to \mathbb{\Delta}_{\mathcal{S}}^{2m+1,m}\,S^n\simeq (\mathrm{Sp}^m\,S^n)\times S^n.$$

Therefore, a model of g_2 is given by $\varphi_2\colon (\wedge(u,v),0)\to(\wedge(x,y,z),0)$, $\varphi_2(u)=x+y$, $\varphi_2(v)=z$, where x, y, z, u and v are of degree n. The previous question gives a model of g_1 as $\varphi_1\colon (\wedge(a,b),0)\to(\wedge(x,y,z),0)$, $\varphi_1(a)=x$, $\varphi_1(b)=y+z$.

By making the change of generators, $u'=u+v$, $v'=v$, $a'=a$, $b'=a+b$, $x'=x$, $y'=x+y+z$, $z'=z$, we get $\varphi_2(u')=y'$, $\varphi_2(v')=z'$, $\varphi_1(a')=x'$ and

$\varphi_1(b') = y'$. This implies that g_1, $g_2 \colon S^n \times S^n \times S^n \to S^n \times S^n$ are respectively the projection on the two first factors and on the two last factors.

Therefore the homotopy pushout $\mathbb{\Delta}_{\mathcal{S}}^{2m+1,m-1} S^n$ has the same rational homotopy type than $S^n \times (S^n * S^n)$.

For a general k, $1 \leq k \leq m-1$, we use a descending induction. The result is already proved for $k = m-1$. For the induction, we consider the homotopy pushout:

$$\begin{tikzcd}
\mathrm{Sp}^k S^n \times \mathbb{\Delta}_{\mathcal{S}}^{2m+1-2k} S^n \arrow[r, "g_1"] \arrow[d, "g_2"'] & \mathrm{Sp}^k S^n \times \mathrm{Sp}^{2m+1-2k} S^n \arrow[d, "g_4"] \\
\mathbb{\Delta}_{\mathcal{S}}^{2m+1,k+1} S^n \arrow[r] & \mathbb{\Delta}_{\mathcal{S}}^{2m+1,k} S^n.
\end{tikzcd}$$

As in the previous case, using the induction hypothesis, we show that this square is rationally homotopic to a homotopy pushout

$$\begin{tikzcd}
S^n \times (S^n \times (*^{m-k-1} S^n)) \arrow[r, "g_1"] \arrow[d, "g_2"'] & S^n \times S^n \arrow[d] \\
S^n \times (*^{m-k-1} S^n) \arrow[r] & \mathbb{\Delta}_{\mathcal{S}}^{2m+1,k} S^n,
\end{tikzcd}$$

where g_1 and g_2 are the product of the identity map with a projection. Thus the homotopy pushout $\mathbb{\Delta}_{\mathcal{S}}^{2m+1,k} S^n$ has the same rational homotopy type than $S^n \times (*^{m-k+1} S^n) \simeq S^n \times S^{(n+1)(m+1-k)-1}$.

(3) Statement (2) implies that the two spaces, $\mathbb{\Delta}_{\mathcal{S}}^{2m-1,k} S^n$ and $\mathbb{\Delta}_{\mathcal{S}}^{2m+1,k+1} S^n$, have the same rational homotopy type. We show that a rational homotopy equivalence between them can be obtained by adding (x_0, x_0) where x_0 is a point of S^n. We prove this result by a descending induction for k, starting from $m-1$ and ending to 1. For $k = m-1$, we consider the commutative diagram

$$\begin{tikzcd}
\mathbb{\Delta}_{\mathcal{S}}^{2m-1,m-1} S^n \arrow[r, "g_1"] & \mathbb{\Delta}_{\mathcal{S}}^{2m+1,m} S^n \\
\mathrm{Sp}^{m-1} S^n \times S^n \arrow[r, "g_2 \times \mathrm{id}"] \arrow[u, "\cong"'] & \mathrm{Sp}^m S^n \times S^n, \arrow[u, "\cong"]
\end{tikzcd}$$

where g_1 is obtained by adding (x_0, x_0) and g_2 by adding x_0. The map g_2 is a rational homotopy equivalence (see Remark 2.7) so is also g_1. For the inductive step, we consider the next cube.

$$\begin{tikzcd}
\mathrm{Sp}^k S^n \times \mathbb{\Delta}_{\mathcal{S}}^{2m-2,k-1} S^n \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Sp}^k S^n \times \mathrm{Sp}^{2m-2,k-1} S^n \arrow[dr] \arrow[dd] & \\
& \mathrm{Sp}^{k+1} S^n \times \mathbb{\Delta}_{\mathcal{S}}^{2m-2,k-1} S^n \arrow[rr] \arrow[dd] & & \mathrm{Sp}^{k+1} S^n \times \mathrm{Sp}^{2m-2,k-1} S^n \arrow[dd] \\
\mathbb{\Delta}_{\mathcal{S}}^{2m-1,k+1} S^n \arrow[rr] \arrow[dr] & & \mathbb{\Delta}_{\mathcal{S}}^{2m-1,k} S^n \arrow[dr] & \\
& \mathbb{\Delta}_{\mathcal{S}}^{2m+1,k+2} S^n \arrow[rr] & & \mathbb{\Delta}_{\mathcal{S}}^{2m+1,k+1}
\end{tikzcd}$$

By Proposition 3.2, the front and the back faces are homotopy pushouts. The morphisms between these two squares are represented by oblique arrows in the previous diagram. The oblique arrows on the top consist to add the point x_0 and are rational homotopy equivalences. The oblique arrows on the bottom are the adding of (x_0, x_0). The result follows by induction. $\square$

4. Finite subsets spaces and Proof of Theorem 1

In this section, we present a generalization of a theorem of Handel concerning the inclusion of a space X in its n-th finite subsets space, $\exp^n X$.

Definition 4.1. Let X be a non-empty space. *The n-th finite subsets space of* X is the space $\exp^n X$ of non-empty subsets of size at most n, topologised as the quotient of X^n by the surjective map $X^n \to \exp^n X$, $(x_1, \dots, x_n) \mapsto \{x_1\} \cup \cdots \cup \{x_n\}$.

If X is pointed, we denote by $\exp^n_* X$ the subspace of $\exp^n X$ formed by the subsets that contain the basepoint. The space $\exp^n_* X$ is pointed by $\{*\}$.

The correspondence $X \mapsto \exp^n X$ is a homotopy functor. C. Tuffley and R.A. Biro have determined its value on some spheres. For $n \geq 2$, we have:

- $\exp^{2k}(S^1) = \exp^{2k-1}(S^1) = S^{2k-1}$, [**26**, Theorem 4],
- $\exp^n(S^2) \simeq_{\mathbb{Q}} S^{2n} \vee S^{2n-2}$, $\exp^n_*(S^2) \simeq_{\mathbb{Q}} S^{2n-2}$,[**29**, Theorem 1],
- $\exp^{2i+1}(S^{2k+1}) \simeq_{\mathbb{Q}} S^{(2k+1)(i+1)+i}$, $\exp^{2i+2}(S^{2k+1}) \simeq_{\mathbb{Q}} S^{(2k+1)(i+1)+i}$, $\exp^n(S^{2k}) \simeq_{\mathbb{Q}} S^{2kn} \vee S^{2k(n-1)}$, [**1**, Lemma 6.3.2. and Lemma 6.3.8.].

Let X be a CW-complex. The space $\exp^n X$ can be described as the following pushout

4.2.
$$\begin{tikzcd} \Delta^n_{\mathcal{S}} X \arrow[r] \arrow[d] & \mathrm{Sp}^n X \arrow[d] \\ \exp^{n-1} X \arrow[r] & \exp^n X. \end{tikzcd}$$

As the map $\Delta^n_{\mathcal{S}} X \to \mathrm{Sp}^n X$ is the inclusion of a sub-CW-complex in a CW-complex, this pushout is also a homotopy pushout. Observe that this pushout is the key in the study of the finite subsets spaces of surfaces done by C. Tuffley in [**29**, Lemma 4].

A *homotopy cofibration* is a sequence $A \xrightarrow{g} X \xrightarrow{j} Y$ where $j \colon X \to Y$ is the homotopy cofiber of $g \colon A \to X$. In the sequel, we do not make any distinction between a map and its homotopy class. The proof of Theorem 1 uses the two next lemmas.

Lemma 4.3. *For any homotopy cofibration of pointed spaces, $A \xrightarrow{g} X \xrightarrow{j} Y$, if the inclusion $\iota^n_X \colon X \to \exp^n_* X$ is homotopically trivial, then the inclusion $\iota^{n+1}_Y \colon Y \to \exp^{n+1}_* Y$ is also homotopically trivial.*

Proof. Any homotopy cofibration $A \xrightarrow{g} X \xrightarrow{j} Y$ gives a coaction of ΣA on Y, denoted by $\nabla \colon Y \to Y \vee \Sigma A$ and defined by pinching the cone CA in $Y \simeq X \cup CA$. For any space T, and any couple of maps $f \colon Y \to T$, $\mu \colon \Sigma A \to T$, we denote by (f, μ) the composition of $f \vee \mu \colon Y \vee \Sigma A \to T \vee T$ with the folding map $T \vee T \to T$.

From two maps f and μ, the coaction gives a map $(f, \mu) \circ \nabla \colon Y \to T$ denoted by f^μ. This correspondence satisfies (cf. [**15**, Chapter 15]) :

$$(f^\mu)^\nu = f^{\mu+\nu} \text{ et } f^* = f,$$

where $* \colon \Sigma A \to T$ is the constant map on the basepoint and ν is a map from ΣA to T. Moreover, one can construct a long exact sequence

$$A \xrightarrow{g} X \xrightarrow{j} Y \xrightarrow{\partial} \Sigma A \xrightarrow{\Sigma g} \Sigma X \longrightarrow \cdots$$

where each pair of consecutive maps is a homotopy cofibration. Therefore, for any space T, we get an exact sequence

$$\cdots \longrightarrow [\Sigma X, T] \xrightarrow{(\Sigma g)_*} [\Sigma A, T] \xrightarrow{\partial_*} [Y, T] \xrightarrow{j_*} [X, T] \xrightarrow{g_*} [A, T].$$

When the maps are homomorphisms of groups, the word *exact* means the usual definition. Between pointed sets, it means that the image of the first application coincides with the preimage of the basepoint by the second one. P. Hilton studies the properties of this exact sequence in [**15**] and proves in particular:

(1) if f and h are elements of $[Y, T]$ such that $j_*(f) = j_*(h)$, then there exists $\mu\colon \Sigma A \to T$ such that $h = f^\mu$, cf. [**15**, 15.5],
(2) the map ∂_* satisfies $\partial_*(\mu) = *^\mu$, thus $(\partial_*(\mu))^\nu = \partial_*(\mu + \nu)$, cf. [**15**, Proof of 15.3],
(3) $\partial_*(\mu) = \partial_*(\nu)$ if, and only if, $\partial_*(\mu - \nu) = *$, cf. [**15**, Proof of 15.6].

We begin now with the proof of the lemma. The commutativity of the next diagram in full lines

$$\begin{tikzcd}
X \arrow[r, "\iota_X^n"] \arrow[d, "j"'] & \exp_*^n X \arrow[d, "\exp_*^n j"] & \\
Y \arrow[r, "\iota_Y^n"'] \arrow[d, "\partial"'] & \exp_*^n Y \arrow[r, hook, "\iota"'] & \exp_*^{n+1} Y \\
\Sigma A \arrow[ur, dashed, "k"'] & &
\end{tikzcd}$$

implies the existence of a map $k\colon \Sigma A \to \exp_*^n Y$ such that $\iota_Y^n = k \circ \partial$. If we compose ι_Y^n with the inclusion $\iota\colon \exp_*^n Y \hookrightarrow \exp_*^{n+1} Y$, we have $\iota_Y^{n+1} = \iota \circ \iota_Y^n = \iota \circ k \circ \partial = \partial_*(\iota \circ k)$. Consider now the commutative diagram:

$$\begin{tikzcd}
 & Y \vee \Sigma A \arrow[r, "\mathrm{id}_Y \vee k"] \arrow[d] & Y \vee \exp_*^n Y \arrow[d] \arrow[dr, "\psi'"] & \\
Y \arrow[ur, "\nabla"] \arrow[r, "\mathrm{id}_Y \times \partial"'] & Y \times \Sigma A \arrow[r, "\mathrm{id}_Y \times k"'] & Y \times \exp_*^n Y \arrow[r, "\psi"'] & \exp_*^{n+1} Y,
\end{tikzcd}$$

where $\psi(y, \Gamma) = \{y\} \cup \Gamma$ and ψ' is the restriction of ψ. Using the previous determination of $\iota \circ \iota_Y^n$, we see that the composition of the maps located on the bottom line is $\iota \circ \iota_Y^n = \partial_*(\iota \circ k)$. We have also

$$\psi' \circ (\mathrm{id}_Y \vee k) \circ \nabla = (\iota \circ \iota_Y^n, \iota \circ k) \circ \nabla = (\iota \circ \iota_Y^n)^{\iota \circ k} = (\partial_*(\iota \circ k))^{\iota \circ k} = \partial_*(2(\iota \circ k)).$$

Therefore, we get $\partial_*(\iota \circ k) = \partial_*(2(\iota \circ k))$ and $\partial_*(\iota \circ k) = *$. Thus $\iota_Y^{n+1} = \iota \circ \iota_Y^n = \partial_*(\iota \circ k) = *$ as required. $\square$

Lemma 4.4. *For any space X, the inclusion $\Sigma X \to \exp_*^3 \Sigma X$ is homotopically trivial.*

Proof. Let $*$ be the basepoint of S^1. Theorem 4.1 of Handel ([**14**]) implies that the inclusion $\exp_*^2 S^1 \cong S^1 \to \exp_*^3 S^1$ induces the zero map between the homotopy groups. Therefore there exists a map

$$G\colon S^1 \times [0,1] \to \exp_*^3 S^1,\ G(v,s) = \{G_1(v,s), G_2(v,s), G_3(v,s)\}$$

such that

- $G(*, s) = \{*\}$, pour tout $s \in [0,1]$,
- $G(v, 0) = \{v\}$, pour tout $v \in S^1$,
- $G(v, 1) = \{*\}$, pour tout $v \in S^1$.

We can now define a homotopy H between the inclusion of $\Sigma X = S^1 \wedge X$ in $\exp_*^3 \Sigma X$ and the constant map by

$$H((v,x),s) = \{G_1(v,s)\wedge x, G_2(v,s)\wedge x, G_3(v,s)\wedge x\}, \text{ for } v \in S^1,\, x \in X,\, s \in [0,1].$$

□

Recall now that the LS-category of a space X is the least integer n such that X can be covered by $(n+1)$ open sets contractible in X. For CW-complexes, an equivalent definition is given using Ganea fibrations $q_n \colon G_n(X) \to X$, of fiber $i_n \colon F_n(X) \to G_n(X)$, defined as follows: q_0 is the path space fibration and q_{n+1} is the fibration associated to the extension $G_n(X) \cup_{i_n} CF_n(X) \to X$ of q_n sending the cone $CF_n(X)$ on the basepoint. In [**12**], Ganea proved that $\mathrm{cat}(X) \leq n$ if, and only if, q_n admits a section. For other equivalent definitions and basic properties of the Lusternik-Schnirelmann category (LS-category), we send the reader to [**6**].

Proof of Theorem 1.

(1) First, we check that the inclusion $G_n(X) \to \exp_*^{n+2} G_n(X)$ is homotopically trivial. This is true for $n = 1$ grants to Lemma 4.4 because $G_1(X) = \Sigma\Omega X$. Suppose now that the result is true for $G_n(X)$. As $G_{n+1}(X)$ is the cofibre of a map with values in $G_n(X)$, the result comes directly from Lemma 4.3.

Suppose now $\mathrm{cat}(X) = n$ and let $r \colon X \to G_n(X)$ be a section of q_n. The commutativity of the next diagram implies the homotopy triviality of the inclusion of X in $\exp_*^{n+2} X$:

$$\begin{tikzcd} X \arrow[r, hook, "\iota_X"] \arrow[d, "r"'] & \exp_*^{n+2} X \\ G_n(X) \arrow[r, hook] & \exp_*^{n+2} G_n(X) \arrow[u, "\exp_*^{n+2} q_n"'] \end{tikzcd}$$

(2) For a space of LS-category n, the diagonal map $\Delta \colon X \to X^{n+1}$ factorizes through the fat wedge. Thus the inclusion of X in $\exp^{n+2} X$ factorizes through $\exp_*^{n+2} X$ and the result follows from (1).

(3) Let $\iota_X^{n+2} \colon X \to \exp_*^{n+2} X$ be the inclusion. The inclusion $\exp_*^{k+1} X \to \exp_*^{(n+1)k+1} X$ factorizes as

$$\exp_*^{k+1} X \xrightarrow{\exp_*^{k+1}(\iota_X^{n+2})} \exp_*^{k+1}(\exp_*^{n+2} X) \xrightarrow{\psi} \exp_*^{k(n+1)+1} X,$$

where the map ψ is the union. The triviality of ι_X^{n+2} implies the result.

□

5. Rational homotopy of n-th finite subsets spaces for $n = 3$

The space $\exp^2 X$ being homeomorphic to $\mathrm{Sp}^2 X$, the first interesting case is $\exp^3 X$.

PROPOSITION 5.1. *Let X be a finite CW-complex. The finite subsets space having at most 3 elements,* $\exp^3 X$*, is the homotopy pushout*

$$\begin{array}{ccc} X \times X & \xrightarrow{g} & \operatorname{Sp}^3 X \\ \Big\downarrow{\scriptstyle \rho_2} & & \Big\downarrow \\ \operatorname{Sp}^2 X & \longrightarrow & \exp^3 X, \end{array}$$

where $g(x,y) = \langle x,x,y\rangle$.

PROOF. We already know that $\exp^3(X)$ is the homotopy pushout

$$\begin{array}{ccc} \mathbb{\Delta}^3_{\mathcal{S}}(X) & \longrightarrow & \operatorname{Sp}^3 X \\ \Big\downarrow & & \Big\downarrow \\ \operatorname{Sp}^2 X & \longrightarrow & \exp^3 X. \end{array}$$

The result comes now from the homeomorphism $X \times X \xrightarrow{\cong} \mathbb{\Delta}^3_{\mathcal{S}} X$, induced by $X \times X \to X \times X \times X$, $(x,y) \mapsto (x,x,y)$. □

If V is a graded vector space, we denote by sV the suspension of V defined by $(sV)^n = V^{n-1}$.

PROPOSITION 5.2. *If ΣX is an r-connected suspension of cohomology $H = H^*(\Sigma X;\mathbb{Q})$, the space $\exp^3 \Sigma X$ is a $(2r+1)$-connected suspension. Its rational cohomology is given by*

$$H^+(\exp^3 \Sigma X;\mathbb{Q}) \cong \wedge^3 H^+ \oplus \wedge^2 H^+ \oplus s\mathrm{Sym}^2(H^+),$$

where $\mathrm{Sym}^2(H^+)$ *is the symmetric power of* H^+.

PROOF. From Proposition 5.1 and classical constructions in rational homotopy theory, a model of $\exp^3 X$ is given by the kernel of a surjective map $\varphi = \overline{\mu} + \overline{\rho}$, where $\overline{\mu}$ and $\overline{\rho}$ are respective models of g and ρ_2. Since the map g is the composition

$$\Sigma X \times \Sigma X \xrightarrow{\Delta \times \mathrm{id}} \Sigma X \times \Sigma X \times \Sigma X \xrightarrow{\rho_3} \operatorname{Sp}^3 \Sigma X,$$

a model of g is given by $\overline{\mu}\colon (\wedge H^+ / \wedge^{>3} H^+, 0) \to (H \otimes H, 0)$, $\overline{\mu}(a) = 2a \otimes 1 + 1 \otimes a$ for $a \in H^+$. A model of $\rho_2 \colon \Sigma X \times \Sigma X \to \operatorname{Sp}^2(\Sigma X)$ has already be made explicit; we modify it in a surjective map $\overline{\mu}$ as follows:

$$\overline{\mu}\colon (A,D) = (\mathrm{Sym}^2(H^+) \oplus s\mathrm{Sym}^2(H^+) \oplus \wedge^2 H, D) \to (H \otimes H, 0),$$

$\overline{\mu}$ is the inclusion on $\mathrm{Sym}^2(H^+) \oplus \wedge^2 H$ and zero on $s\mathrm{Sym}^2(H^+)$. The differential D is given by $D(w) = sw$ for $w \in \mathrm{Sym}^2(H^+)$ and zero on $s\mathrm{Sym}^2(H^+) \oplus \wedge^2 H$. The map $\varphi = \overline{\mu} + \overline{\rho}$ is clearly surjective. If we write an element of $A \oplus (\wedge H^+ / \wedge^{>3} H^+)$ as a couple (a,b), the kernel of φ is the vector space

$$(s\mathrm{Sym}^2(H^+), 0) \oplus (0, \wedge^3 H^+) \oplus \left\{(x,0) - 2(0,x) \mid x \in \wedge^2 H^+\right\}.$$

The product and the differential being null, the space $\exp^3 \Sigma X$ is a rational suspension. All the elements in this kernel having a degree greater than or equal to $2r+2$, the space $\exp^3 \Sigma X$ is rationally $(2r+1)$-connected. □

6. Rational homotopy of n-th finite subsets spaces for $n = 4$

PROPOSITION 6.1. *Let X be a finite CW-complex. The finite subsets space having at most 4 elements is the homotopy pushout*

$$\begin{tikzcd} X \times \operatorname{Sp}^2 X \arrow[r, "g_2'"] \arrow[d, "g_1'"'] & \operatorname{Sp}^4 X \arrow[d] \\ \operatorname{Sp}^3 X \arrow[r] & \exp^4 X, \end{tikzcd}$$

where $g_1'(x, \langle x_1, x_2\rangle) = \langle x, x_1, x_2\rangle$ and $g_2'(x, \langle x_1, x_2\rangle) = \langle x, x, x_1, x_2\rangle$.

PROOF. Consider the next diagram, where the map $\mathbb{\Delta}^4_{\mathcal{S}}(X) \to \exp^3 X$ is induced by $X \times \operatorname{Sp}^2 X \to \operatorname{Sp}^3 X$, $(x, \langle x_1, x_2\rangle) \mapsto \langle x, x_1, x_2\rangle$.

$$\begin{tikzcd} X \times X \arrow[r, "g_1"] \arrow[d, "g_2"'] & X \times \operatorname{Sp}^2 X \arrow[r] \arrow[d, "g_4"'] & \operatorname{Sp}^3 X \arrow[d] \\ \operatorname{Sp}^2(X) \arrow[r, "g_3"'] & \mathbb{\Delta}^4_{\mathcal{S}}(X) \arrow[r] \arrow[d] & \exp^3 X \arrow[d] \\ & \operatorname{Sp}^4 X \arrow[r] & \exp^4 X \end{tikzcd}$$

The bottom square is a homotopy pushout by definition of $\exp^4 X$. The top rectangle is a homotopy pushout by Proposition 5.1 and the top left square also by Proposition 3.2. Then the top right square and the square of the statement are homotopy pushouts from classical manipulations with homotopy pushouts. □

PROPOSITION 6.2. *If ΣX is an r-connected suspension of cohomology $H = H^*(\Sigma X; \mathbb{Q})$, the space $\exp^4 \Sigma X$ is a $(2r+1)$-connected suspension.*

PROOF. A model for the map $g_2' \colon X \times \operatorname{Sp}^2(\Sigma X) \to \operatorname{Sp}^4(\Sigma X)$ is the morphism $\overline{\mu} \colon (\wedge H^+ / \wedge^{>4} H^+, 0) \to (H \otimes \wedge^2 H, 0)$, $\overline{\mu}(a) = 2a \otimes 1 + 1 \otimes a$.

We choose an ordered basis of H^+ and denote by W the subvector space of $H \otimes \wedge^2 H$ generated by the elements $a \otimes b$, with $a < b$, the elements $a \otimes a$, with a of odd degree, the elements $a \otimes bc$ with $a < b$ or $a < c$ and the elements $a \otimes ac$ with a of odd degree. We then form a surjective cdga model $\overline{\rho}$ of g_1' as follows:

$$\overline{\rho} \colon (A, D) = (\wedge^3 H \oplus W \oplus dW, D) \to (H \otimes \wedge^2 H, 0),$$

with all the products on A null except $(a \otimes b) \cdot c = (a \otimes bc)$, and $d(a \otimes b) \cdot c = d(a \otimes bc)$ for $a \otimes b \in W$, $c \in H^+$ and $a \otimes bc \in W$. The differential D is defined by $D(w) = dw$ if $w \in W$ and zero otherwise. The map $\overline{\rho}$ is the injection on W and is zero on dW. Moreover $\overline{\rho}(a) = a \otimes 1 + 1 \otimes a$ for $a \in H^+$. As the sum

$$\varphi = (\overline{\rho} + \overline{\mu}) : (A, D) \oplus (\wedge^4 H, 0) \to (H \otimes \wedge^2 H, 0)$$

is a surjective map, its kernel is a model for $\exp^4 \Sigma X$. This kernel is the sum $(dW, 0) \oplus (0, \wedge^4 H^+) \oplus \{2(x, 0) - (0, x)\,, x \in \wedge^3 H^+\}$. The product and the differential being trivial on this kernel, the space $\exp^4 \Sigma X$ is a suspension. All the elements in this kernel having a degree greater than or equal to $2r + 2$, the space $\exp^4 \Sigma X$ is rationally $(2r + 1)$-connected. □

References

1. Ross A. Biro, *The structure of cyclic configuration spaces and the homotopy type of Kontsevich's orientation space*, Ph.D. thesis, Stanford, 1994.
2. Karol Borsuk, *On the third symmetric potency of the circumference*, Fund. Math. **36** (1949), 236–244. MR MR0035987 (12,42a)
3. Karol Borsuk and Stanislaw Ulam, *On symmetric products of topological spaces*, Bull. Amer. Math. Soc. **37** (1931), no. 12, 875–882. MR MR1562283
4. Raoul Bott, *On the third symmetric potency of* S^1, Fund. Math. **39** (1952), 264–268 (1953). MR MR0054954 (14,1003e)
5. Glen E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972, Pure and Applied Mathematics, Vol. 46. MR MR0413144 (54 #1265)
6. Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré, *Lusternik-Schnirelmann category*, Mathematical Surveys and Monographs, vol. 103, American Mathematical Society, Providence, RI, 2003. MR MR1990857 (2004e:55001)
7. Albrecht Dold and René Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. (2) **67** (1958), 239–281. MR MR0097062 (20 #3542)
8. J. L. Dupont and G. Lusztig, *On manifolds satisfying* $w_1\mathrm{Sp}2 = 0$, Topology **10** (1971), 81–92. MR MR0273631 (42 #8508)
9. Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR MR1802847 (2002d:55014)
10. Yves Félix, John Oprea, and Daniel Tanré, *Algebraic models in geometry*, Oxford Graduate Texts in Mathematics, vol. 17, Oxford University Press, Oxford, 2008.
11. Yves Félix and Daniel Tanré, *The cohomology algebra of unordered configuration spaces*, J. London Math. Soc. (2) **72** (2005), no. 2, 525–544. MR MR2156668 (2006d:55017)
12. Tudor Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois J. Math. **11** (1967), 417–427. MR MR0229240 (37 #4814)
13. Karsten Grove, Stephen Halperin, and Micheline Vigué-Poirrier, *The rational homotopy theory of certain path spaces with applications to geodesics*, Acta Math. **140** (1978), no. 3-4, 277–303. MR MR496895 (80g:58024)
14. David Handel, *Some homotopy properties of spaces of finite subsets of topological spaces*, Houston J. Math. **26** (2000), no. 4, 747–764. MR MR1823966 (2002d:55016)
15. Peter Hilton, *Homotopy theory and duality*, Gordon and Breach Science Publishers, New York, 1965. MR MR0198466 (33 #6624)
16. Sadok Kallel and Denis Sjerve, *Finite subset spaces and a spectral sequence of Biro*, 2008.
17. ———, *Remarks on finite subset spaces*, 2009.
18. I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Cambridge Philos. Soc. **58** (1962), 563–568. MR MR0143204 (26 #764)
19. Jacob Mostovoy, *Geometry of truncated symmetric products and real roots of real polynomials*, Bull. London Math. Soc. **30** (1998), no. 2, 159–165. MR MR1489327 (99a:57021)
20. ———, *Lattices in* $\mathbb{C}$ *and finite subsets of a circle*, Amer. Math. Monthly **111** (2004), no. 4, 357–360. MR MR2057192
21. Boon W. Ong, *The homotopy type of the symmetric products of bouquets of circles*, Internat. J. Math. **14** (2003), no. 5, 489–497. MR MR1993792 (2004g:55010)
22. S. C. F. Rose, *A hyperbolic approach to* $\exp^3 S^1$, 2007.
23. V. P. Snaith and J. J. Ucci, *Three remarks on symmetric products and symmetric maps*, Pacific J. Math. **45** (1973), 369–377. MR MR0367988 (51 #4230)
24. Dennis Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269–331 (1978). MR MR0646078 (58 #31119)
25. Daniel Tanré, *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, Lecture Notes in Mathematics, vol. 1025, Springer-Verlag, Berlin, 1983. MR MR764769 (86b:55010)
26. Christopher Tuffley, *Finite subset spaces of* S^1, Algebr. Geom. Topol. **2** (2002), 1119–1145 (electronic). MR MR1998017 (2004f:54008)
27. ———, *Finite subset spaces of graphs and punctured surfaces*, Algebr. Geom. Topol. **3** (2003), 873–904 (electronic). MR MR2012957 (2004i:55023)
28. ———, *Connectivity of finite subset spaces of cell complexes*, Pacific J. Math. **217** (2004), no. 1, 175–179. MR MR2105772 (2005g:55019)

29. ______, *Finite subset spaces of closed surfaces*, 2007.
30. Don Bernard Zagier, *Equivariant Pontrjagin classes and applications to orbit spaces. Applications of the G-signature theorem to transformation groups, symmetric products and number theory*, Lecture Notes in Mathematics, Vol. 290, Springer-Verlag, Berlin, 1972. MR MR0339202 (49 #3965)

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